

On Mathon's construction of maximal arcs in Desarguesian planes

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Dedicated to Adriano Barlotti on the occasion of his 80th birthday

Abstract. We study the problem of determining the largest d of a non-Denniston maximal arc of degree 2^d generated by a $\{p, 1\}$ -map in $\text{PG}(2, 2^m)$ via a recent construction of Mathon [9]. On one hand, we show that there are $\{p, 1\}$ -maps that generate non-Denniston maximal arcs of degree $2^{(m+1)/2}$, where $m \geq 5$ is odd. Together with Mathon's result [9] in the m even case, this shows that there are always $\{p, 1\}$ -maps generating non-Denniston maximal arcs of degree $2^{\lfloor (m+2)/2 \rfloor}$ in $\text{PG}(2, 2^m)$. On the other hand, we prove that the largest degree of a non-Denniston maximal arc in $\text{PG}(2, 2^m)$ constructed using a $\{p, 1\}$ -map is less than or equal to 2^{m-3} . We conjecture that this largest degree is actually $2^{\lfloor (m+2)/2 \rfloor}$ when $m > 9$.

Key words. Arc, linearized polynomial, maximal arc, quadratic form.

1 Introduction

Let $\text{PG}(2, q)$ be the Desarguesian projective plane of order q , q a prime power. A set of k points in $\text{PG}(2, q)$ is called a (k, n) -arc if no $n + 1$ points of the set are collinear. The number n is usually called the *degree* of the arc.

Let \mathcal{K} be a (k, n) -arc in $\text{PG}(2, q)$, and let P be a point in \mathcal{K} . Then each of the $q + 1$ lines through P contains at most $n - 1$ points of \mathcal{K} . Therefore

$$k \leq 1 + (q + 1)(n - 1) = qn + n - q.$$

A (k, n) -arc is said to be *maximal* if $k = qn + n - q$. From the above argument, it is easily seen that any line of $\text{PG}(2, q)$ that contains a point of a maximal arc \mathcal{K} must contain exactly n points of that arc; that is

$$|L \cap \mathcal{K}| = 0 \text{ or } n$$

for every line L of $\text{PG}(2, q)$. Therefore the degree n of a maximal $(qn + n - q, n)$ -arc must divide q .

The study of arcs of degree greater than two was started by Barlotti [2]. For $q = 2^m$, Denniston [3] constructed maximal $(qn + n - q, n)$ -arcs in $\text{PG}(2, q)$ for every

$n, n \mid q, n < q$ (see also [6, p. 304]). Thas [10], [11] also gave two other constructions of maximal arcs of certain degrees in $\text{PG}(2, 2^m)$, where m is even. For odd prime powers q , Ball, Blokhuis and Mazzocca [1] proved that maximal arcs of degree n do not exist in $\text{PG}(2, q)$, when $n < q$. Recently Mathon [9] gave a new construction of maximal arcs in $\text{PG}(2, 2^m)$ that generalizes the construction of Denniston. We give a brief account of his construction.

Let \mathcal{C} be the set of all conics

$$F_{\alpha, \beta, \lambda} = \{(x, y, z) \in \text{PG}(2, 2^m) \mid \alpha x^2 + xy + \beta y^2 + \lambda z^2 = 0\}$$

where $\alpha, \beta \in \mathbb{F}_{2^m}^*$ and $\alpha x^2 + x + \beta$ is irreducible over \mathbb{F}_{2^m} (that is, $\text{Tr}_{2^m/2}(\alpha\beta) = 1$, here $\text{Tr}_{2^m/2}$ is the trace map from \mathbb{F}_{2^m} to \mathbb{F}_2). For $\lambda, \lambda' \in \mathbb{F}_{2^m}$, $\lambda \neq \lambda'$ we define a composition

$$F_{\alpha, \beta, \lambda} \oplus F_{\alpha', \beta', \lambda'} = F_{\alpha \oplus \alpha', \beta \oplus \beta', \lambda + \lambda'}$$

where

$$a \oplus a' = \frac{a\lambda + a'\lambda'}{\lambda + \lambda'} \quad \text{for any } a, a' \in \mathbb{F}_{2^m}.$$

A subset \mathcal{F} of \mathcal{C} is said to be *closed* under the composition \oplus if for any $F_1, F_2 \in \mathcal{F}$ with $F_1 \neq F_2$ we have $F_1 \oplus F_2 \in \mathcal{F}$. In [9] Mathon proved that the set of points of all conics in a closed set of conics together with the common nucleus $F_0 = F_{\alpha, \beta, 0} = (0, 0, 1)$ forms a maximal arc in $\text{PG}(2, 2^m)$. When all conics in a closed set of conics come from a single pencil of conics, Mathon's construction gives rise to Denniston maximal arcs. In general, Mathon showed that closed sets of conics can be obtained by using linearized polynomials over \mathbb{F}_{2^m} . Specifically, Mathon proved the following theorem.

Theorem 1.1 ([9, Theorem 2.5]). *Let $p(x) = \sum_{i=0}^{d-1} a_i x^{2^i-1}$ and $q(x) = \sum_{i=0}^{d-1} b_i x^{2^i-1}$ be polynomials with coefficients in \mathbb{F}_{2^m} . For an additive subgroup A of order 2^d in \mathbb{F}_{2^m} let $\mathcal{F} = \{F_{p(\lambda), q(\lambda), \lambda} \mid \lambda \in A \setminus \{0\}\} \subset \mathcal{C}$ be a set of conics with common nucleus F_0 . If $\text{Tr}_{2^m/2}(p(\lambda)q(\lambda)) = 1$ for every $\lambda \in A \setminus \{0\}$, then the set of points on all conics in \mathcal{F} together with F_0 forms a maximal $(2^{m+d} - 2^m + 2^d, 2^d)$ -arc \mathcal{K} in $\text{PG}(2, 2^m)$. If both $p(x), q(x)$ have $d \leq 2$, then \mathcal{K} is a Denniston arc.*

Hamilton [4] gave the following test for when the arc \mathcal{K} in Theorem 1.1 is a Denniston arc.

Theorem 1.2 ([4, Theorem 2.1]). *Let $p(x)$ and $q(x)$ be the same polynomials as given in Theorem 1.1, let A be an additive subgroup of size 2^d in \mathbb{F}_{2^m} , and let \mathcal{K} be the maximal arc obtained in Theorem 1.1. Then \mathcal{K} is of Denniston type if and only if for all $\lambda, \lambda' \in A \setminus \{0\}$, $\lambda \neq \lambda'$, both $(p(\lambda) + p(\lambda'))/(\lambda + \lambda')$ and $(q(\lambda) + q(\lambda'))/(\lambda + \lambda')$ are constant.*

Mathon posed several problems at the end of his paper [9]. The third problem he posed is: What is the largest d of a non-Denniston maximal arc of degree 2^d generated by a $\{p, q\}$ -map in $\text{PG}(2, 2^m)$ via Theorem 1.1? When m is even, Mathon [9] showed that there exists a non-Denniston maximal arc of degree $2^{m/2+1}$ generated by a $\{p, 1\}$ -map in $\text{PG}(2, 2^m)$. When m is odd, Hamilton [4] showed that there exists a non-Denniston maximal arc of degree 8 generated by a $\{p, 1\}$ -map in $\text{PG}(2, 2^m)$, where $m \geq 5$. In this paper, we concentrate on the following restricted version of Mathon's problem: What is the largest d of a non-Denniston maximal arc of degree 2^d generated by a $\{p, 1\}$ -map in $\text{PG}(2, 2^m)$ via Theorem 1.1? In Section 2, we show that there are $\{p, 1\}$ -maps that generate non-Denniston maximal arcs of degree $2^{(m+1)/2}$, where $m \geq 5$ is odd. Together with Mathon's result [9, Theorem 3.2] in the m even case, this shows that there are always $\{p, 1\}$ -maps generating non-Denniston maximal arcs of degree $2^{\lfloor (m+2)/2 \rfloor}$ in $\text{PG}(2, 2^m)$. In Section 3 we prove that if a maximal arc generated by a $\{p, 1\}$ -map via Theorem 1.1 has degree 2^{m-1} or 2^{m-2} and $m \geq 7$, then it is a Denniston maximal arc. Hence when $m \geq 7$, the largest degree of a non-Denniston maximal arc constructed using a $\{p, 1\}$ -map via Theorem 1.1 is less than or equal to 2^{m-3} . We conjecture that when $m > 9$, this largest degree is actually $2^{\lfloor (m+2)/2 \rfloor}$ and provide some evidence for this conjecture.

2 Maximal arcs in $\text{PG}(2, 2^m)$, m odd

In this section m is always an odd positive integer, and γ always denotes an element of \mathbb{F}_{2^m} with $\text{Tr}_{2^m/2}(\gamma) = 1$. To simplify notation, from now on, we will use Tr in place of $\text{Tr}_{2^m/2}$ if there is no confusion. We start with the following lemma.

Lemma 2.1. *Let $S_\gamma = \{x \in \mathbb{F}_{2^m} \mid \text{Tr}(\gamma x + x^3) = 0\}$. Then there exists a choice of $\gamma \in \mathbb{F}_{2^m}$ such that S_γ contains an \mathbb{F}_2 -subspace A with $\dim(A) = \frac{m+1}{2}$.*

Proof. Let $Q_\gamma(x) = \text{Tr}(\gamma x + x^3)$ and let $V = \mathbb{F}_{2^m}$. The map $Q_\gamma : V \rightarrow \mathbb{F}_2$ is a quadratic form on V over \mathbb{F}_2 . The corresponding bilinear form B is given by $B(x, y) = Q_\gamma(x + y) - Q_\gamma(x) - Q_\gamma(y) = \text{Tr}(x^2 y + x y^2)$, hence

$$\begin{aligned} \text{Rad } V &= \{x \in V \mid B(x, y) = 0 \text{ for each } y \in V\} \\ &= \{x \in V \mid \text{Tr}(x^2 y + x y^2) = 0 \text{ for each } y \in V\} \\ &= \{x \in V \mid \text{Tr}(y(x^2 + \sqrt{x})) = 0 \text{ for each } y \in V\} \\ &= \{x \in V \mid x^2 = \sqrt{x}\}. \end{aligned}$$

Since m is odd, we conclude that $\text{Rad } V = \mathbb{F}_2$. Note that in characteristic 2, the quadratic form $Q_\gamma(x)$ is not necessarily zero on $\text{Rad } V$. Therefore we define

$$V_0 = \{x \in \text{Rad } V \mid Q_\gamma(x) = 0\}.$$

This is an \mathbb{F}_2 -space of dimension equal to $\dim(\text{Rad } V)$ or $\dim(\text{Rad } V) - 1$. Since $\text{Tr}(\gamma) = 1$, we have $V_0 = \text{Rad } V = \mathbb{F}_2$. Hence $\text{rank}(Q_\gamma) = m - 1$ is even and Q_γ is

either hyperbolic or elliptic. It is always possible to choose $\gamma \in \mathbb{F}_{2^m}$, with $\text{Tr}(\gamma) = 1$, such that Q_γ is hyperbolic on V/V_0 . (This can be seen from the weight distribution of the dual of the double-error-correcting BCH code, see [8, p. 451]). With this choice of γ , the maximum dimension of a subspace of V/V_0 on which Q_γ vanishes is $\frac{m-1}{2}$. Let U be such a subspace and let $A = U \perp V_0$. Then $\dim(A) = \frac{m+1}{2}$ and $Q_\gamma(x)$ vanishes on A , hence $A \subset S_\gamma$. This completes the proof. \square

Now let $\gamma \in \mathbb{F}_{2^m}$ be chosen such that $\text{Tr}(\gamma) = 1$ and $S_\gamma = \{x \in \mathbb{F}_{2^m} \mid \text{Tr}(\gamma x + x^3) = 0\}$ contains an \mathbb{F}_2 -subspace A of \mathbb{F}_{2^m} of dimension $\frac{m+1}{2}$. Let $p(x) = 1 + \gamma x + x^3$. Then we have the following corollary of Theorem 1.1.

Theorem 2.2. *The set of points on the conics $\mathcal{F} = \{F_{p(\lambda),1,\lambda} \mid \lambda \in A \setminus \{0\}\}$ together with the common nucleus F_0 forms a maximal arc \mathcal{K} in $\text{PG}(2, 2^m)$ of degree $2^{(m+1)/2}$. When $m \geq 5$, the maximal arc \mathcal{K} is non-Denniston.*

Proof. Let $p(\lambda) = 1 + \gamma\lambda + \lambda^3$, with the choice of γ as above, and let A be the $(\frac{m+1}{2})$ -dimensional \mathbb{F}_2 -subspace in S_γ given by Lemma 2.1. Then we have $\text{Tr}(p(\lambda)) = \text{Tr}(1) = 1$ for every $\lambda \in A \setminus \{0\}$. By Theorem 1.1, the first part of the theorem follows.

When $m \geq 5$, the maximal arc \mathcal{K} is non-Denniston. This can be seen as follows. For $\lambda, \lambda' \in A \setminus \{0\}$, $(p(\lambda) + p(\lambda'))/(\lambda + \lambda') = \gamma + \lambda^2 + \lambda\lambda' + \lambda'^2$. When $|A| \geq 8$, this expression cannot be constant when $\lambda, \lambda', \lambda \neq \lambda'$, run through $A \setminus \{0\}$. Therefore by Theorem 1.2, the arc \mathcal{K} is not of Denniston type. \square

Theorem 2.2 together with Mathon's result ([9, Theorem 3.2]) in the m even case shows that there are always $\{p, 1\}$ -maps generating non-Denniston maximal arcs of degree $2^{\lfloor (m+2)/2 \rfloor}$ in $\text{PG}(2, 2^m)$, when $m \geq 5$.

3 Some upper bounds on the degree of non-Denniston maximal arcs from $\{p, 1\}$ -maps

We start this section by making some remarks about Theorem 1.1. In Theorem 1.1, Mathon restricted the degrees of the polynomials $p(\lambda)$, $q(\lambda)$ to be less than or equal to $2^{d-1} - 1$, where the subspace $A \subset \mathbb{F}_{2^m}$ involved has size 2^d . We will show that there is no loss of generality in doing so.

Proposition 3.1. *Let $f(x) = \sum_{i=0}^{m-1} a_i x^{2^i-1} \in \mathbb{F}_{2^m}[x]$, and let A be an \mathbb{F}_2 -subspace in \mathbb{F}_{2^m} of size 2^d , where $d \leq m-1$. Then there exists a polynomial $f_1(x) = \sum_{i=0}^{d-1} b_i x^{2^i-1} \in \mathbb{F}_{2^m}[x]$ such that $f(\lambda) = f_1(\lambda)$ for every $\lambda \in A \setminus \{0\}$.*

Proof. Let $A(x) = \prod_{\lambda \in A} (x - \lambda)$. This is a degree 2^d linearized polynomial in $\mathbb{F}_{2^m}[x]$ (see [7, p. 110], also [8, p. 119]), that is,

$$A(x) = x^{2^d} + c_{d-1}x^{2^{d-1}} + \cdots + c_0x,$$

where $c_i \in \mathbb{F}_{2^m}$. Let $a(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_0$. The polynomials $A(x)$ and $a(x)$ are called 2-associates of each other (see [7, p. 115]). Let $f(x) = G(x)/x$, where $G(x) =$

$\sum_{i=0}^{m-1} a_i x^{2^i}$, and let $g(x) = \sum_{i=0}^{m-1} a_i x^i$ be the 2-associate of $G(x)$. Using the division algorithm, we write

$$g(x) = k(x)a(x) + r(x), \quad (3.1)$$

where $\deg r(x) < \deg a(x) = d$. Let $K(x)$ and $R(x)$ be the 2-associates of $k(x)$ and $r(x)$ respectively. Turning (3.1) into linearized 2-associates, and noting that the 2-associate of $k(x)a(x)$ is $K(A(x))$, the composition of $A(x)$ with $K(x)$ (cf. [7, p. 115], Lemma 3.59), we get

$$G(x) = K(A(x)) + R(x), \quad (3.2)$$

with $\deg R(x) \leq 2^{d-1}$. With $f_1(x) = R(x)/x$, we see from (3.2) that $f(\lambda) = f_1(\lambda)$ for every $\lambda \in A \setminus \{0\}$. \square

We note that if one does not restrict the degree of the polynomials $p(x)$, $q(x)$ to be less than or equal to $2^{d-1} - 1$ (where $2^d = |A|$), Theorem 1.1 still holds, but then it sometimes leads to Denniston maximal arcs, which, at first sight, may not look like Denniston. We give a couple of examples of this situation below. So by restricting the degrees of the polynomials $p(x)$, $q(x)$ to be less than or equal to $2^{d-1} - 1$ in Theorem 1.1, not only is there no loss of generality (by Proposition 3.1), but also some “trivial” examples are avoided.

Example 3.2. Let $p(x) = a_0 + \frac{x+x^2+x^4+\dots+x^{2^{m-1}}}{x} \in \mathbb{F}_{2^m}[x]$, where $\text{Tr}(a_0) = 1$. Let $A = \{x \in \mathbb{F}_{2^m} \mid \text{Tr}(x) = 0\}$. Then we have $\text{Tr}(p(\lambda)) = 1$ for every $\lambda \in A \setminus \{0\}$. This $p(x)$ indeed gives rise to a maximal arc of degree 2^{m-1} in $\text{PG}(2, 2^m)$ by Mathon's construction. But the maximal arc in this example is of Denniston type by Theorem 1.2 since for every $\lambda \in A \setminus \{0\}$, we have $p(\lambda) = a_0$, a constant.

Example 3.3. Let $p(x) = \sum_{i=0}^{m-1} a_i x^{2^i-1} \in \mathbb{F}_{2^m}[x]$, where $\text{Tr}(a_0) = 1$. We may choose $a_1, a_2, \dots, a_{m-1} \in \mathbb{F}_{2^m}$ such that $A = \{\lambda \in \mathbb{F}_{2^m} \mid a_1 \lambda^2 + a_2 \lambda^{2^2} + \dots + a_{m-1} \lambda^{2^{m-1}} = 0\}$ has dimension $m-2$ over \mathbb{F}_2 . Then we have $\text{Tr}(p(\lambda)) = 1$ for every $\lambda \in A \setminus \{0\}$. This $p(x)$ gives rise to a maximal arc of degree 2^{m-2} in $\text{PG}(2, 2^m)$ by Mathon's construction. But the maximal arc in this example is again of Denniston type by Theorem 1.2 since for every $\lambda \in A \setminus \{0\}$, $p(\lambda) = a_0$, a constant.

Next we prove that when $m \geq 5$ the largest d of a non-Denniston maximal arc of degree 2^d generated by a $\{p, 1\}$ -map via Theorem 1.1 is less than $m-1$.

Theorem 3.4. *Let A be an additive subgroup of size 2^{m-1} in \mathbb{F}_{2^m} , where $m \geq 5$. Let $p(x) = \sum_{i=0}^{m-2} a_i x^{2^i-1} \in \mathbb{F}_{2^m}[x]$. If $\text{Tr}(p(\lambda)) = 1$ for all $\lambda \in A \setminus \{0\}$, then $a_2 = a_3 = \dots = a_{m-2} = 0$, thus $p(x)$ is linear and the maximal arc obtained via Theorem 1.1 is of Denniston type.*

Proof. Every hyperplane in \mathbb{F}_{2^m} can be written as $\{x \in \mathbb{F}_{2^m} \mid \text{Tr}(ax) = 0\}$ for some nonzero $a \in \mathbb{F}_{2^m}$. By making a change of variable in $p(x)$, we may assume that $A = \{x \in \mathbb{F}_{2^m} \mid \text{Tr}(x) = 0\}$. We consider two cases.

Case 1: $\text{Tr}(a_0) = 1$. In this case, if $\text{Tr}(p(\lambda)) = 1$ for all $\lambda \in A \setminus \{0\}$, then $\text{Tr}(\sum_{i=1}^{m-2} a_i \lambda^{2^i-1}) = 0$ for all $\lambda \in A \setminus \{0\}$. Thus, $(1 + \text{Tr}(x)) \text{Tr}(\sum_{i=1}^{m-2} a_i x^{2^i-1})$, viewed as a function from \mathbb{F}_{2^m} to itself, is identically zero. That is, in $\mathbb{F}_{2^m}[x]$, we have

$$(1 + \text{Tr}(x)) \cdot \text{Tr}\left(\sum_{i=1}^{m-2} a_i x^{2^i-1}\right) \equiv 0 \pmod{x^{2^m} - x} \quad (3.3)$$

Let $t(x) = \text{LHS of (3.3)} = (1 + x + x^2 + \cdots + x^{2^m-1}) \text{Tr}(\sum_{i=1}^{m-2} a_i x^{2^i-1})$.

Claim: The coefficient of $x^{2^m-2^r+2}$ in $t(x)$ is $a_{m-r}^{2^r} + a_{m-r+1}^{2^r}$ for $3 \leq r \leq m-2$.

The m -bit binary representation of $2^m - 2^r + 2$ is

$$\underbrace{1 \dots 1}_{m-r \geq 2} \underbrace{0 \dots 0}_{r-2 \geq 1} 10,$$

which contains two blocks of 1's (separated by 0's). (We will always number the bits from right to left as $0, 1, 2, \dots, m-1$.) Note that the exponents of the summands in $1 + \text{Tr}(x)$, written in m -bit binary representation, are $000 \dots 000$, $000 \dots 001$, $000 \dots 010, \dots, 100 \dots 000$, and the exponents of the summands in $\text{Tr}(\sum_{i=1}^{m-2} a_i x^{2^i-1})$ are cyclic shifts of

$$000 \dots 001, 000 \dots 011, 000 \dots 0111, 000 \dots 01111, \dots \quad \text{and} \quad \underbrace{00}_2 \underbrace{11 \dots 111}_{m-2}.$$

When we multiply $1 + \text{Tr}(x)$ with $\text{Tr}(\sum_{i=1}^{m-2} a_i x^{2^i-1})$, there are two ways to obtain $x^{2^m-2^r+2}$, namely adding the exponent of a summand in $1 + \text{Tr}(x)$ to the exponent of a summand in $\text{Tr}(\sum_{i=1}^{m-2} a_i x^{2^i-1})$ with or without carry.

Suppose that we are in the latter case. The exponent from $1 + \text{Tr}(x)$ must be 2 while the exponent from $\text{Tr}(\sum_{i=1}^{m-2} a_i x^{2^i-1})$ is a shift of $2^{m-r} - 1$.

$$\underbrace{1 \dots 1}_r 0 \dots 010 = 0 \dots 010 + \underbrace{1 \dots 1}_{m-r \geq 2} \underbrace{0 \dots 0}_r$$

Thus, this case contributes the coefficient $a_{m-r}^{2^r}$.

Now suppose that we are in the former case. Since bit-1 of $2^m - 2^r + 2$ is 1 while bit-0 is 0, the exponent $2^m - 2^r + 2$ must be obtained as 2^0 added to $(2^m - 2^r) + (2^1 - 2^0)$:

$$\underbrace{1 \dots 1}_{m-r \geq 2} 0 \dots 010 = 0 \dots 01 + \underbrace{1 \dots 1}_{m-r \geq 2} 0 \dots 01,$$

so this case contributes the coefficient $a_{m-r+1}^{2^r}$. The claim now follows. In particular, by (3.3), we find that $a_2 = a_3 = \cdots = a_{m-2}$.

Claim: The coefficient of x^{2^m-4} in $t(x)$ is $a_{m-2}^4 + a_{m-3}^4 + a_{m-3}^8$.

Clearly the exponent $(2^m - 4) = 11 \dots 100$ can be obtained by

$$11 \dots 100 = 00 \dots 000 + 11 \dots 100$$

This contributes the coefficient a_{m-2}^4 .

Also the exponent $2^m - 4$ can be obtained by adding a non-zero exponent in $1 + \text{Tr}(x)$ to an exponent from $\text{Tr}(\sum_{i=2}^{m-2} a_i x^{2^i-1})$. Suppose that when adding the exponents, there is no carry. We have two ways to obtain $2^m - 4$, namely,

$$11 \dots 100 = 10 \dots 0 + 011 \dots 100,$$

$$11 \dots 100 = 0 \dots 0100 + 11 \dots 1000.$$

This contributes the coefficient $a_{m-3}^4 + a_{m-3}^8$. Finally we note that there is no way of getting $2^m - 4$ as a sum of exponents inducing a carry. Thus, the coefficient of x^{2^m-4} in $t(x)$ is as claimed. This implies $a_{m-3} = 0$, which yields $a_2 = a_3 = \dots = a_{m-2} = 0$. Hence $p(\lambda) = a_0 + a_1 \lambda$.

Case 2: $\text{Tr}(a_0) = 0$. We have $\text{Tr}(\sum_{i=1}^{m-2} a_i \lambda^{2^i-1}) = 1$ for all $\lambda \in A \setminus \{0\}$. Hence $(1 + \text{Tr}(x)) \cdot (1 + \text{Tr}(\sum_{i=1}^{m-2} a_i x^{2^i-1}))$, viewed as a function from \mathbb{F}_{2^m} to itself, is the characteristic function of the subset $\{0\}$ of \mathbb{F}_{2^m} . Therefore,

$$(1 + \text{Tr}(x)) + (1 + \text{Tr}(x)) \cdot \text{Tr}\left(\sum_{i=1}^{m-2} a_i x^{2^i-1}\right) \equiv 1 - x^{2^m-1} \pmod{x^{2^m} - x} \quad (3.4)$$

Note that the binary representation of the exponent of x^{2^m-1} is $111 \dots 1$ (m ones altogether), while in the left hand side of (3.4), the binary representation of the exponent of any term in the product $(1 + \text{Tr}(x)) \cdot \text{Tr}(\sum_{i=1}^{m-2} a_i x^{2^i-1})$ cannot have more than $1 + (m-2) = m-1$ ones. So (3.4) cannot hold. Thus, this case does not occur. This completes our proof. \square

Remarks. (1) Theorem 3.4 is not true when $m = 4$. In $\text{PG}(2, 16)$, there exists a degree 8 non-Denniston maximal arc (cf. Section 4.1 of [9]).

(2) It is interesting to note that when $m \geq 5$ a non-Denniston maximal arc of degree 2^{m-1} (i.e., the dual of a hyperoval) in $\text{PG}(2, 2^m)$ can be obtained from $\{p, q\}$ -maps via Theorem 1.1, with $q(x) \neq 1$. See [9, p. 362] for an example in $\text{PG}(2, 32)$. Theorem 3.4 shows that this cannot be achieved if $m \geq 5$ and $q(x)$ is restricted to be 1.

The ideas in the proof of Theorem 3.4 can be further used to prove the following theorem. The proof contains more complicated computations.

Theorem 3.5. *Let A be an additive subgroup of size 2^{m-2} in \mathbb{F}_{2^m} , where $m \geq 7$. Let $p(x) = \sum_{i=0}^{m-3} a_i x^{2^i-1} \in \mathbb{F}_{2^m}[x]$. If $\text{Tr}(p(\lambda)) = 1$ for all $\lambda \in A \setminus \{0\}$ then $a_2 = a_3 = \dots = a_{m-3} = 0$, thus $p(x)$ is linear and the maximal arc obtained via Theorem 1.1 is of Denniston type.*

Proof. Since A has dimension $m - 2$ over \mathbb{F}_2 , we may assume that $A = \{x \in \mathbb{F}_{2^m} \mid \text{Tr}(x) = 0 \text{ and } \text{Tr}(\mu x) = 0\}$ for some $\mu \in \mathbb{F}_{2^m}^*$ with $\mu \neq 1$. Again we consider two cases.

Case 1: $\text{Tr}(a_0) = 1$. Then

$$(1 + \text{Tr}(x))(1 + \text{Tr}(\mu x)) \text{Tr}\left(\sum_{i=1}^{m-3} a_i x^{2^i-1}\right) \equiv 0 \pmod{x^{2^m} - x} \quad (3.5)$$

$$(1 + \text{Tr}(x) + \text{Tr}(\mu x) + \text{Tr}(x) \text{Tr}(\mu x)) \cdot \text{Tr}\left(\sum_{i=1}^{m-3} a_i x^{2^i-1}\right) \equiv 0 \pmod{x^{2^m} - x}$$

Let $r(x)$ denote the LHS of (3.5), $s(x) = 1 + \text{Tr}(x) + \text{Tr}(\mu x) + \text{Tr}(x) \text{Tr}(\mu x)$, and $t(x) = \text{Tr}(\sum_{i=1}^{m-3} a_i x^{2^i-1})$. The exponent of each term in $r(x)$ is a sum of the exponent of a summand in $s(x)$ and the exponent of some summand in $t(x)$. Similar to the proof of Theorem 3.4, exponents of the summands in $t(x)$ are $2^i - 1$, $1 \leq i \leq m - 3$, and their cyclic shifts. Exponents from $s(x)$ are 0 ; 2^i ; and $2^i + 2^j$, $i \neq j$. The terms x^0 , x^{2^i} , and $x^{2^i+2^j}$ ($i \neq j$) in $s(x)$ have coefficients 1 , $1 + \mu^{2^i} + \mu^{2^{i-1}}$, and $\mu^{2^i} + \mu^{2^j}$, respectively.

Claim: The coefficient of $x^{(2^m-1)-2^{m-2}-2^{m-4}}$ in $r(x)$ is

$$a_{m-3}^{2^{m-1}}(1 + \mu^{2^{m-3}} + \mu^{2^{m-4}}) + a_{m-4}(\mu^{2^{m-1}} + \mu^{2^{m-3}}) \\ + a_{m-4}^{2^{m-1}}(\mu^{2^{m-3}} + \mu^{2^{m-5}}) + a_{m-3}(\mu^{2^{m-1}} + \mu^{2^{m-4}}).$$

The binary representation of the exponent of any term in $r(x)$ cannot have more than $2 + (m - 3) = m - 1$ ones. The binary expansion of $(2^m - 1) - 2^{m-2} - 2^{m-4}$ is $101011 \dots 1$. This involves $m - 2$ ones, so it can be obtained as a sum of two exponents (one from $s(x)$, the other from $t(x)$) without carry or with *exactly* one carry. Assume that we are in the former case. There are only three ways to obtain $(2^m - 1) - 2^{m-2} - 2^{m-4}$, namely,

$$101011 \dots 1 = 001000 \dots 0 + 100011 \dots 1 \\ = 101000 \dots 0 + 000011 \dots 1 \\ = 001010 \dots 0 + 100001 \dots 1.$$

These contribute the coefficient $(1 + \mu^{2^{m-3}} + \mu^{2^{m-4}})a_{m-3}^{2^{m-1}} + (\mu^{2^{m-1}} + \mu^{2^{m-3}})a_{m-4} + (\mu^{2^{m-3}} + \mu^{2^{m-5}})a_{m-4}^{2^{m-1}}$ for $x^{(2^m-1)-2^{m-2}-2^{m-4}}$ in $r(x)$. (Here we used the assumption that $m \geq 7$. If $m = 5$, the coefficient of the term $x^{2^4+2^2+1}$ in $r(x)$ is not the same as in our claim. The reason is that, for example, $10101 = 00100 + 10001$ leads to another possibility, namely 00100 comes from $a_4^4 x^4$ in $t(x)$, and 10001 comes from $(\mu^{2^0} + \mu^{2^4})x^{2^0+2^4}$ in $s(x)$. This cannot happen if $m \geq 7$.)

Now assume that a carry had been induced. The last carry-over must have occurred either at bit- $(m-3)$ or bit- $(m-1)$. The latter case cannot occur.

$$10101 \dots 1 = 10010 \dots 0 + 0001 \dots 1.$$

This contributes the coefficient $(\mu^{2^{m-1}} + \mu^{2^{m-4}})a_{m-3}$. This proves the claim. By (3.5), we have

$$\begin{aligned} & a_{m-3}^{2^{m-1}}(1 + \mu^{2^{m-3}} + \mu^{2^{m-4}}) + a_{m-4}(\mu^{2^{m-1}} + \mu^{2^{m-3}}) \\ & + a_{m-4}^{2^{m-1}}(\mu^{2^{m-3}} + \mu^{2^{m-5}}) + a_{m-3}(\mu^{2^{m-1}} + \mu^{2^{m-4}}) = 0 \end{aligned} \quad (3.6)$$

Claim: The coefficient of $x^{(2^m-1)-2^{m-1}-2^{m-4}}$ is

$$a_{m-4}(\mu^{2^{m-2}} + \mu^{2^{m-3}}) + a_{m-3}(\mu^{2^{m-2}} + \mu^{2^{m-4}}).$$

The binary expansion of $(2^m - 1) - 2^{m-1} - 2^{m-4}$ is 011011...1. Suppose it is obtained as a sum of exponents from $s(x)$ and $t(x)$ without carry. Then

$$01101 \dots 1 = 0110 \dots 0 + 00001 \dots 1$$

which contributes $(\mu^{2^{m-2}} + \mu^{2^{m-3}})a_{m-4}$. (Here again we have used the assumption that $m \geq 7$. If $m = 6$, the coefficient of the term $x^{2^4+2^3+2+1}$ in $r(x)$ is not the same as in our claim. The reason is that $011011 = 011000 + 000011$ leads to another possibility, namely 011000 comes from $a_2^8 x^{2^4+2^3}$ in $t(x)$, and 000011 comes from $(\mu^{2^0} + \mu^2)x^{2^0+2}$ in $s(x)$. This cannot happen if $m \geq 7$.)

If $(2^m - 1) - 2^{m-1} - 2^{m-4}$ is obtained as a sum of exponents from $s(x)$ and $t(x)$ with a carry, the last carry-over must occur at bit- $(m-2)$ or bit-0.

$$01101 \dots 1 = 01010 \dots 00 + 00011 \dots 11.$$

This contributes the coefficient $(\mu^{2^{m-2}} + \mu^{2^{m-4}})a_{m-3}$. Therefore the claim is proved, and by (3.5), we have

$$a_{m-4}(\mu^{2^{m-2}} + \mu^{2^{m-3}}) = a_{m-3}(\mu^{2^{m-2}} + \mu^{2^{m-4}}). \quad (3.7)$$

Claim: $a_{m-3}^{2^{m-1}}(1 + \mu^{2^{m-3}} + \mu^{2^{m-4}}) + a_{m-4}^{2^{m-1}}(\mu^{2^{m-3}} + \mu^{2^{m-5}}) = 0$.

The claim is equivalent to

$$a_{m-3}(1 + \mu^{2^{m-2}} + \mu^{2^{m-3}}) + a_{m-4}(\mu^{2^{m-2}} + \mu^{2^{m-4}}) = 0.$$

Consider the expression

$$\begin{aligned} E &= (a_{m-3}(1 + \mu^{2^{m-2}} + \mu^{2^{m-3}}) + a_{m-4}(\mu^{2^{m-2}} + \mu^{2^{m-4}}))(\mu^{2^{m-2}} + \mu^{2^{m-3}}) \\ &= a_{m-3}(\mu^{2^{m-2}} + \mu^{2^{m-3}}) + a_{m-3}(\mu^{2^{m-2}} + \mu^{2^{m-3}})^2 \\ &\quad + a_{m-4}(\mu^{2^{m-2}} + \mu^{2^{m-4}})(\mu^{2^{m-2}} + \mu^{2^{m-3}}). \end{aligned}$$

Using (3.7), we have

$$\begin{aligned} E &= a_{m-3}(\mu^{2^{m-2}} + \mu^{2^{m-3}}) + a_{m-3}(\mu^{2^{m-1}} + \mu^{2^{m-2}}) + a_{m-3}(\mu^{2^{m-2}} + \mu^{2^{m-4}})^2 \\ &= 0. \end{aligned}$$

Since $\mu \neq 0, 1$ we have $(\mu^{2^{m-2}} + \mu^{2^{m-3}}) \neq 0$ and our claim follows. In particular, by (3.6) it implies $a_{m-4}(\mu^{2^{m-1}} + \mu^{2^{m-3}}) = a_{m-3}(\mu^{2^{m-1}} + \mu^{2^{m-4}})$. Adding this to (3.7) we get

$$a_{m-4}(\mu^{2^{m-1}} + \mu^{2^{m-2}}) = a_{m-3}(\mu^{2^{m-1}} + \mu^{2^{m-2}}).$$

Hence $a_{m-4} = a_{m-3}$. Substituting a_{m-4} in (3.7) by a_{m-3} , we have $a_{m-3} = 0$.

Claim: Let $m-4 > k > 2$. If $a_j = 0$ for all $m-3 > j > k$ then $a_k = 0$.

We will use a similar argument to that in (3.7). To this end we consider the coefficient of $x^{(2^k-1)+2^{m-2}+2^{m-3}}$ in $r(x)$. The binary expansion of its exponent is 0110...01...1. This includes $2+k$ ones. All a_j with $j > k$ are zero. The sum of an exponent from $1 + \text{Tr}(x) + \text{Tr}(\mu x) + \text{Tr}(x)\text{Tr}(\mu x)$ and an exponent from $\text{Tr}(\sum_{i=0}^k a_i x^{2^i-1})$ has at most $2+k$ ones. Since $k > 2$ there is only one way to obtain $(2^k-1) + 2^{m-2} + 2^{m-3}$, namely,

$$0110\dots 0 \underbrace{1\dots 1}_{k>2} = 0110\dots 0 + 0\dots 0 \underbrace{1\dots 1}_{k>2}.$$

It follows that $(\mu^{2^{m-2}} + \mu^{2^{m-3}})a_k = 0$. Hence $a_k = 0$.

Since $a_{m-3} = a_{m-4} = 0$ we find that $a_3 = \dots = a_{m-4} = a_{m-3} = 0$ by induction.

Claim: $a_2 = 0$.

Consider the coefficients of x^{2^4+7} and x^{2^5+7} . Since for all $j > 2$ we have $a_j = 0$ there are only two ways to obtain each exponent.

$$\begin{aligned} 0\dots 0010111 &= 0\dots 0010100 + 0\dots 0000011 \\ &= 0\dots 0010001 + 0\dots 0000110 \\ 0\dots 0100111 &= 0\dots 0100100 + 0\dots 0000011 \\ &= 0\dots 0100001 + 0\dots 0000110. \end{aligned}$$

Hence the coefficient of x^{2^4+7} is $(\mu^4 + \mu^{16})a_2 + (\mu + \mu^{16})a_2^2$ and the coefficient of x^{2^5+7} is $(\mu^4 + \mu^{32})a_2 + (\mu + \mu^{32})a_2^2$. Adding both values we find

$$a_2(\mu^{16} + \mu^{32}) + a_2^2(\mu^{16} + \mu^{32}) = 0.$$

Thus, a_2 is either 0 or 1. Now look at the coefficient of x^{15} . There are only three ways of obtaining 15 as a sum with the exponents we can use.

$$\begin{aligned}
0 \dots 01111 &= 0 \dots 01100 + 0 \dots 00011 \\
&= 0 \dots 01001 + 0 \dots 00110 \\
&= 0 \dots 00011 + 0 \dots 01100.
\end{aligned}$$

Hence $(\mu^4 + \mu^8)a_2 + (\mu + \mu^8)a_2^2 + (\mu + \mu^2)a_2^4 = 0$. If $a_2 = 1$ then $\mu^2 + \mu^4 = 0$ which is a contradiction. Thus, $a_2 = 0$.

It follows that $a_2 = \dots = a_{m-3} = 0$.

Case 2: $\text{Tr}(a_0) = 0$. Then $\text{Tr}(\sum_{i=1}^{m-3} a_i \lambda^{2^i-1}) = 1$ for all $\lambda \in A \setminus \{0\}$. Hence if we view

$$(1 + \text{Tr}(x) + \text{Tr}(\mu x) + \text{Tr}(x) \text{Tr}(\mu x)) \cdot \left(1 + \text{Tr}\left(\sum_{i=1}^{m-3} a_i x^{2^i-1}\right)\right)$$

as a function from \mathbb{F}_{2^m} to itself, it is the characteristic function of the subset $\{0\}$ of \mathbb{F}_{2^m} . Therefore,

$$\begin{aligned}
&(1 + \text{Tr}(x) + \text{Tr}(\mu x) + \text{Tr}(x) \text{Tr}(\mu x)) + (1 + \text{Tr}(x) + \text{Tr}(\mu x) + \text{Tr}(x) \text{Tr}(\mu x)) \\
&\cdot \text{Tr}\left(\sum_{i=1}^{m-3} a_i x^{2^i-1}\right) \equiv 1 - x^{2^m-1} \pmod{x^{2^m} - x}.
\end{aligned} \tag{3.8}$$

Note that the binary representation of the exponent of x^{2^m-1} is $111 \dots 1$ (m ones altogether), while in the left hand side of (3.8), the binary representation of the exponent of any term in the product

$$(1 + \text{Tr}(x) + \text{Tr}(\mu x) + \text{Tr}(x) \text{Tr}(\mu x)) \cdot \text{Tr}\left(\sum_{i=1}^{m-3} a_i x^{2^i-1}\right)$$

cannot have more than $2 + (m-3) = m-1$ ones. So (3.8) cannot hold. Thus, this case does not occur. This completes our proof. \square

Combining Theorem 3.4 and Theorem 3.5 with the constructive result in Section 2 and Theorem 3.2 in [9], we find that when $m \geq 7$, the largest d of a non-Denniston maximal arc of degree 2^d in $\text{PG}(2, 2^m)$ generated by a $\{p, 1\}$ -map via Theorem 1.1 satisfies

$$\left\lfloor \frac{m+2}{2} \right\rfloor \leq d \leq m-3.$$

We have the following conjecture.

Conjecture 3.6. *When $m > 9$, the largest d of a non-Denniston maximal arc of degree 2^d in $\text{PG}(2, 2^m)$ generated by a $\{p, 1\}$ -map via Theorem 1.1 is $\lfloor \frac{m+2}{2} \rfloor$.*

In order to prove the above conjecture, it suffices to prove the following. Let A be an additive subgroup in \mathbb{F}_{2^m} of size 2^d , where $m > 9$, $p(x) = a_0 + a_1x + \cdots + a_{d-1}x^{2^{d-1}-1} \in \mathbb{F}_{2^m}[x]$. If $d \geq \lfloor \frac{m+2}{2} \rfloor + 1$, $\text{Tr}(p(\lambda)) = 1$ for every $\lambda \in A \setminus \{0\}$, then $a_2 = a_3 = \cdots = a_{d-1} = 0$. So far we can only prove some partial results in this direction.

Theorem 3.7. *Let A be an additive subgroup in \mathbb{F}_{2^m} of size 2^d , where $d \leq m-1$, and let $p(x) = a_0 + a_1x + \cdots + a_{d-2}x^{2^{d-2}-1} \in \mathbb{F}_{2^m}[x]$, with $a_{d-2} \neq 0$. If $\text{Tr}(p(\lambda)) = 1$ for every $\lambda \in A \setminus \{0\}$, then $d \leq \frac{m+2}{2}$.*

Proof. Assume to the contrary that $d > \frac{m+2}{2}$; we will show that $a_{d-2} = 0$. Assume that the defining equation for A is

$$(1 + \text{Tr}(\mu_1 x))(1 + \text{Tr}(\mu_2 x)) \cdots (1 + \text{Tr}(\mu_{m-d} x)) = 1,$$

where $\mu_i \in \mathbb{F}_{2^m}$, $i = 1, 2, \dots, m-d$, are linearly independent over \mathbb{F}_2 . We consider two cases:

Case 1: $\text{Tr}(a_0) = 1$. Then

$$(1 + \text{Tr}(\mu_1 x))(1 + \text{Tr}(\mu_2 x)) \cdots (1 + \text{Tr}(\mu_{m-d} x)) \cdot \text{Tr}\left(\sum_{i=1}^{d-2} a_i x^{2^i-1}\right) \equiv 0 \pmod{x^{2^m} - x}. \quad (3.9)$$

Claim: The coefficient of $x^{1+2+2^2+\cdots+2^{d-3}+2^{d-1}+2^d+\cdots+2^{m-2}}$ is

$$\left(\sum_{\sigma \in S_{m-d}} \mu_{\sigma(1)}^{2^{m-2}} \mu_{\sigma(2)}^{2^{m-3}} \cdots \mu_{\sigma(m-d)}^{2^{d-1}} \right) a_{d-2},$$

where S_{m-d} is the symmetric group on $m-d$ letters.

The exponent of $x^{1+2+2^2+\cdots+2^{d-3}+2^{d-1}+2^d+\cdots+2^{m-2}}$ has m -bit binary representation

$$\underbrace{011\dots1}_{m-d} \underbrace{011\dots1}_{d-2}.$$

Since $d > \frac{m+2}{2}$, we see that $d-2 > m-d$, there is only one way to get the term $x^{1+2+2^2+\cdots+2^{d-3}+2^{d-1}+2^d+\cdots+2^{m-2}}$ when multiplying $(1 + \text{Tr}(\mu_1 x))(1 + \text{Tr}(\mu_2 x)) \cdots (1 + \text{Tr}(\mu_{m-d} x))$ with $\text{Tr}(\sum_{i=1}^{d-2} a_i x^{2^i-1})$, namely

$$\underbrace{011\dots1}_{m-d} \underbrace{011\dots1}_{d-2} = \underbrace{000\dots0}_{m-d} \underbrace{011\dots1}_{d-2} + \underbrace{011\dots1}_{m-d} \underbrace{000\dots0}_{d-2}.$$

Therefore the claim follows. By (3.9), we see that

$$\left(\sum_{\sigma \in S_{m-d}} \mu_{\sigma(1)}^{2^{m-2}} \mu_{\sigma(2)}^{2^{m-3}} \cdots \mu_{\sigma(m-d)}^{2^{d-1}} \right) a_{d-2} = 0. \quad (3.10)$$

Now set $r = m - d$. Then

$$\sum_{\sigma \in S_{m-d}} \mu_{\sigma(1)}^{2^{m-2}} \mu_{\sigma(2)}^{2^{m-3}} \cdots \mu_{\sigma(m-d)}^{2^{d-1}} = \left(\sum_{\sigma \in S_r} \mu_{\sigma(1)}^{2^{r-1}} \mu_{\sigma(2)}^{2^{r-2}} \cdots \mu_{\sigma(r)}^{2^{d-1}} \right)^{2^{d-1}}.$$

Note that

$$\sum_{\sigma \in S_r} \mu_{\sigma(1)}^{2^{r-1}} \mu_{\sigma(2)}^{2^{r-2}} \cdots \mu_{\sigma(r)}^{2^{d-1}} = \det \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_r \\ \mu_1^2 & \mu_2^2 & \cdots & \mu_r^2 \\ & & \cdots & \\ \mu_1^{2^{r-1}} & \mu_2^{2^{r-1}} & \cdots & \mu_r^{2^{r-1}} \end{pmatrix}.$$

We will use $\Delta(\mu_1, \mu_2, \dots, \mu_r)$ to denote this last determinant. Since μ_i , $i = 1, 2, \dots, r$, are linearly independent over \mathbb{F}_2 , we see that $\Delta(\mu_1, \mu_2, \dots, \mu_r) \neq 0$ (cf. [7, p. 109]). By (3.10), this shows that $a_{d-2} = 0$.

Case 2: $\text{Tr}(a_0) = 0$. As before, this case can be easily seen not to occur.

This completes the proof. □

In order to extend the result in Theorem 3.7, we need to introduce more notation. Let $\mu_1, \mu_2, \dots, \mu_r$ be elements in \mathbb{F}_{2^m} that are linearly independent over \mathbb{F}_2 . Let $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq m - 1$ be integers. We define

$$T(\alpha_1, \alpha_2, \dots, \alpha_r) = \sum_{\sigma \in S_r} \mu_{\sigma(1)}^{2^{\alpha_1}} \mu_{\sigma(2)}^{2^{\alpha_2}} \cdots \mu_{\sigma(r)}^{2^{\alpha_r}}.$$

Using the above notation, we have the following lemma.

Lemma 3.8. *Let $m > 9$ be an odd integer, let $r = \frac{m-3}{2}$, and let t be an integer such that $3 \leq t \leq \frac{m-1}{2}$. Then there exist $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq m - 1$ such that*

- (i) $\alpha_r \leq m - t - 3$,
- (ii) $T(\alpha_1, \alpha_2, \dots, \alpha_r) \neq 0$, and
- (iii) *the number of consecutive integers in the set $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ is less than or equal to $t - 1$.*

We postpone the proof of this lemma to the appendix. With this lemma, we can prove the following theorem.

Theorem 3.9. Let $m > 9$ be an odd integer, let A be an additive subgroup in \mathbb{F}_{2^m} of size 2^d , where $d \leq m-1$, and let $p(x) = a_0 + a_1x + a_2x^3 + \cdots + a_tx^{2^t-1} \in \mathbb{F}_{2^m}[x]$, with $a_t \neq 0$ and $t \leq (d-1)$. If $3 \leq t \leq \frac{m-1}{2}$, and $\text{Tr}(p(\lambda)) = 1$ for every $\lambda \in A \setminus \{0\}$, then $d \leq \frac{m+1}{2}$.

Proof. Assume to the contrary that $d > \frac{m+1}{2}$; we will show that $a_t = 0$. Without loss of generality, assume that $d = \frac{m+3}{2}$, and let $r = m - d = \frac{m-3}{2}$. Assume that the defining equation for A is

$$(1 + \text{Tr}(\mu_1 x))(1 + \text{Tr}(\mu_2 x)) \cdots (1 + \text{Tr}(\mu_r x)) = 1,$$

where $\mu_i \in \mathbb{F}_{2^m}$, $i = 1, 2, \dots, r$, are linearly independent over \mathbb{F}_2 . As in the proof of Theorem 3.7, we only need to consider the case where $\text{Tr}(a_0) = 1$. Hence we have

$$(1 + \text{Tr}(\mu_1 x))(1 + \text{Tr}(\mu_2 x)) \cdots (1 + \text{Tr}(\mu_r x)) \text{Tr}\left(\sum_{i=1}^t a_i x^{2^i-1}\right) \equiv 0 \pmod{x^{2^m} - x}. \quad (3.11)$$

By Lemma 3.8, there exist $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq m-1$ such that

- (i) $\alpha_r \leq m - t - 3$,
- (ii) $T(\alpha_1, \alpha_2, \dots, \alpha_r) \neq 0$, and
- (iii) the number of consecutive integers in the set $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ is less than or equal to $t-1$.

We will look at the coefficient of $x^{1+2^2+\cdots+2^r+2^{m-2}+2^{m-3}+\cdots+2^{m-t-1}}$ in the left hand side of (3.11). Note that the exponent of this monomial has the m -bit binary representation

$$\underbrace{011\dots1}_{t} \underbrace{00\dots1\dots1\dots1}_{m-t-2},$$

where at the α_i th bit there is a 1, for each $i = 1, 2, \dots, r$.

Since the number of consecutive integers in the set $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ is less than or equal to $t-1$, there is only one way to get the term

$$x^{1+2^2+\cdots+2^r+2^{m-2}+2^{m-3}+\cdots+2^{m-t-1}}$$

when multiplying

$$(1 + \text{Tr}(\mu_1 x))(1 + \text{Tr}(\mu_2 x)) \cdots (1 + \text{Tr}(\mu_r x)) \quad \text{with} \quad \text{Tr}\left(\sum_{i=1}^t a_i x^{2^i-1}\right),$$

namely

$$\underbrace{011\dots1}_{t} \underbrace{00\dots1\dots1\dots1}_{m-t-2} = \underbrace{000\dots0}_{t} \underbrace{00\dots1\dots1\dots1}_{m-t-2} + \underbrace{011\dots1}_{t} \underbrace{000\dots0}_{m-t-2}.$$

Therefore, the coefficient of $x^{1+2^2+\cdots+2^r+2^{m-2}+2^{m-3}+\cdots+2^{m-t-1}}$ in the left hand side of (3.11) is

$$\left(\sum_{\sigma \in S_r} \mu_{\sigma(1)}^{2^{\alpha_1}} \mu_{\sigma(2)}^{2^{\alpha_2}} \cdots \mu_{\sigma(r)}^{2^{\alpha_r}} \right) a_t^{2^{m-t-1}} = T(\alpha_1, \dots, \alpha_r) a_t^{2^{m-t-1}}.$$

By (3.11), we see that $T(\alpha_1, \dots, \alpha_r) a_t^{2^{m-t-1}} = 0$. Since $T(\alpha_1, \dots, \alpha_r) \neq 0$, we have $a_t = 0$. This completes the proof. \square

4 Appendix

In this appendix, we give a proof of Lemma 3.8. First, we introduce some notation. Let x_1, \dots, x_r be elements in \mathbb{F}_{2^m} that are linearly independent over \mathbb{F}_2 . For any integer i , we set $v_i = (x_1^{2^i}, \dots, x_r^{2^i})$. We use $v_i^{2^j}$ to denote component-wise exponentiation of v_i by 2^j . Hence $v_i^{2^j} = v_{i+j}$. Since $x_\ell^{2^m} = x_\ell$ for all $\ell = 1, 2, \dots, r$, we have $v_m = v_0$. So in what follows, the indices of v_i are to be read modulo m . Now condition (ii) of Lemma 3.8 is equivalent to the vectors

$$v_{\alpha_1}, \dots, v_{\alpha_r}$$

being linearly independent over \mathbb{F}_{2^m} , i.e.,

$$\det \begin{pmatrix} x_1^{2^{\alpha_1}} & \cdots & x_r^{2^{\alpha_1}} \\ \vdots & \ddots & \vdots \\ x_1^{2^{\alpha_r}} & \cdots & x_r^{2^{\alpha_r}} \end{pmatrix} \neq 0.$$

Let V be the \mathbb{F}_{2^m} -span of v_0, \dots, v_{m-1} . By [7, Lemma 3.51], $\dim_{\mathbb{F}_{2^m}} V = r$ and $\{v_i, v_{i+1}, \dots, v_{i+r-1}\}$ is an \mathbb{F}_{2^m} -basis of V for any $0 \leq i \leq m-r$.

In the following, we will be considering subspaces of V spanned by some vectors in $\{v_0, v_1, \dots, v_{m-1}\}$. To this end, we will use binary vectors to represent subsets of $\{v_0, \dots, v_{m-1}\}$. Let $\mathbf{u} = (u_0, u_1, \dots, u_{i-1})$ be a vector with entries in $\{0, 1\}$. Then the subset of $\{v_0, v_1, \dots, v_{m-1}\}$ represented by \mathbf{u} is

$$S(\mathbf{u}) = \{v_\ell \mid u_\ell \neq 0, 0 \leq \ell \leq i-1\}.$$

By $V(\mathbf{u})$ we will denote the \mathbb{F}_{2^m} -span of the vectors in $S(\mathbf{u})$. For example, if $\mathbf{u} = (1, 1, 0, 1)$ then $V(\mathbf{u}) = \mathbb{F}_{2^m} v_0 + \mathbb{F}_{2^m} v_1 + \mathbb{F}_{2^m} v_3$. For convenience, we also allow concatenation of binary vectors. If $\mathbf{u} = (u_0, u_1, \dots, u_{i-1})$ and $\mathbf{u}' = (u'_0, \dots, u'_{j-1})$ then the concatenation of \mathbf{u} with \mathbf{u}' is

$$\mathbf{u} * \mathbf{u}' = (u_0, \dots, u_{i-1}, u'_0, \dots, u'_{j-1}).$$

Moreover $\underbrace{\mathbf{u} * \mathbf{u} * \cdots * \mathbf{u}}_\ell$ is abbreviated to $\mathbf{u}^{*\ell}$.

Now we can reformulate Lemma 3.8 as follows: For every integer t such that $3 \leq t \leq \frac{m-1}{2}$, there exists a binary vector \mathbf{u} of length at most $m - (t+2)$ such that $V(\mathbf{u}) = V$ and the number of consecutive 1's in \mathbf{u} is at most $t-1$. It is this reformulation that we will prove in this appendix.

One final preparation before we give the proof. Given integers i and $j > 0$, let $I(i, j)$ denote the \mathbb{F}_{2^m} -span of $v_i, v_{i+1}, \dots, v_{i+j-1}$. Given a subspace W of V , we define $W^{2^t} = \{w^{2^t} \mid w \in W\}$, where w^{2^t} means component-wise exponentiation of w by 2^t . We will need the following lemma.

Lemma 4.1. *Suppose that $W = V(\mathbf{u})$ where $\mathbf{u} = (u_0, u_1, \dots, u_{s-1}) \in \mathbb{F}_2^s$. If $I(s, t) \subset W$ and $W^{2^t} \cap I(0, s) \subset W$ then $W = V$.*

Proof. By assumption, W is spanned by a subset of $\{v_0, \dots, v_{s-1}\}$. Let $v_i \in W$, $0 \leq i \leq s-1$, be one of the generating vectors. If $i+t \leq s-1$, then $v_i^{2^t} = v_{i+t} \in I(0, s) \cap W^{2^t} \subset W$. If $i+t > s-1$, then $v_i^{2^t} = v_{i+t} \in I(s, t) \subset W$. Hence for any vector $v_i \in W$, $0 \leq i \leq s-1$, we have $v_{i+t} \in W$. Extending this property to linear combinations of the generating vectors of W , we see that $I(s+t, t) \subset W$ since $I(s, t) \subset W$. That is, $I(s+\ell t, t) \subset W$ for all $\ell \geq 0$. Hence $v_i \in W$ for all $0 \leq i \leq m-1$ and $W = V$. \square

Proof of Lemma 3.8. Write $r = kt + a$ where $0 \leq a \leq t-1$. Since $r = \frac{m-3}{2}$, we have $m = 2kt + 2a + 3$. Set $\mathbf{a} = (1, 1, \dots, 1) \in \mathbb{F}_2^a$ and $\mathbf{u} = (0, 1, \dots, 1) \in \mathbb{F}_2^t$. Let

$$V(i) = V(\mathbf{a} * \mathbf{u}^{*i}).$$

That is, $V(i)$ is the space spanned by the vectors in $S(\mathbf{a} * \mathbf{u}^{*i})$. Then $V(k)$ is the \mathbb{F}_{2^m} -span of $\{v_0, v_1, \dots, v_{r-1}\} \setminus \{v_a, v_{a+t}, \dots, v_{a+(k-1)t}\}$. Since $\{v_0, v_1, \dots, v_{r-1}\}$ is a basis for V , we see that $\dim V(k) = r - k$. Let b be the smallest nonnegative integer such that $V(k+b) = V(k+b+1)$. In particular, $V(k+i)$ is a proper subspace of $V(k+i+1)$ if $0 \leq i < b$. We observe that $0 \leq b \leq k$. There are three cases to consider.

Case 1: $\dim V(k+1) \geq r - k + 2$. In this case $b \leq k-1$. If $V(k+b) = V$, then $S(\mathbf{a} * \mathbf{u}^{*(k+b)})$ spans V . Note that $\mathbf{a} * \mathbf{u}^{*(k+b)}$ has length $a + (k+b)t \leq a + (2k-1)t = m - a - (t+3)$. By construction this vector does not have more than $t-1$ consecutive 1's. So we are done in this case.

If $V(k+b) \neq V$ then $b \leq k-2$. Let $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{F}_2^t$ and $\mathbf{a}' = (1, 0, \dots, 0) \in \mathbb{F}_2^t$. Let

$$\mathbf{w}_i = \mathbf{a} * \mathbf{u}^{*(k+b)} * \mathbf{0} * \mathbf{a}'^{*i}.$$

We define $W(i) = V(\mathbf{w}_i)$ to be the \mathbb{F}_{2^m} -span of the vectors in $S(\mathbf{w}_i)$. In particular, $W(0) = V(k+b)$. Let b' be the smallest nonnegative integer such that $W(b') = W(b'+1)$.

$$\begin{aligned} \dim W(0) &= \dim V(k+b) \\ &\geq \dim V(k+1) + (b-1) \\ &\geq r - k + 2 + b - 1 \\ &= r - (k - b - 1). \end{aligned}$$

Hence $0 \leq b' \leq k - 1 - b$. We claim that

- (i) $W(b') \supset I(a + (b + k + i + 1)t, t)$
- (ii) $W(b') \supset W(b')^{2^t} \cap I(0, a + (b + k + i + 1)t)$

for all $i \geq 0$. By Lemma 4.1, these two claims imply that $W(b') = V$. The length of $w_{b'}$ is

$$a + (k + b + 1 + b')t \leq a + 2kt = m - a - 3.$$

Note that the last $t - 1$ entries in $w_{b'}$ are zero. Dropping these $t - 1$ positions we obtain a vector of length $m - a - (t + 2)$. This vector does not have more than $t - 1$ consecutive 1's and it corresponds to a subset of $\{v_{a_1}, \dots, v_{a_r}\}$ that spans V , hence Lemma 3.8 is proved in this case once we prove the above two claims.

To prove the first claim, we recall that $W(b') = V(a * u^{*(k+b)} * 0 * a'^{*b'})$. Hence $v_{a+(k+b+1+i)t} \in W(b')$ for all $i \geq 0$ since this vector corresponds to the first position in the i -th copy of a' . Now $W(b') \supseteq V(k + b) = V(k + b + 1 + i)$ for all $i \geq 0$, we also have $S(a * u^{*(k+b+1+i)}) \subset W(b')$. Thus

$$v_{a+(k+b+1+i)t+1}, \dots, v_{a+(k+b+1+i)t+(t-1)} \in W(b'),$$

since these vectors correspond to the nonzero positions in the last copy of u in $a * u^{*(k+b+2+i)}$. This proves our first claim.

For the second claim it suffices to show that $S(0 * a * u^{*(k+b)} * 0 * a'^{*i}) \subseteq W(b')$. Hence we need to show that the vectors corresponding to the $(k + b)$ -th copy of u and the i -th copy of a' , respectively, are in $W(b')$. The former is true since $W(b')$ includes $S(a * u^{*(k+b+1)})$ which spans $V(k + b + 1)$. The latter holds because $W(b') = W(b' + 1)$ which includes the vectors in $S(a * u^{*(k+b)} * 0 * a'^{*i(b'+1)})$. This proves our second claim.

Case 2: $\dim V(k + 1) = r - k + 1 = \dim V(k) + 1$. In this case, one of the vectors $v_{r+1}, \dots, v_{r+(t-1)}$ does not belong to $V(k)$. Suppose that vector is $v_{r+j} = v_{a+kt+j}$, $1 \leq j \leq t - 1$. Then $V(k + 1) = V(k) + \mathbb{F}_{2^m} v_{r+j}$. Since any linear dependence relation translates to a linear dependence relation when both sides are raised to the 2^t th power, we get $\dim V(k + i) \leq \dim V(k) + i$, $i \geq 0$.

Subcase 1: $V(k + b) \neq V$, i.e., $b < k$. As seen above, all vectors $v_{r+1}, \dots, v_{r+(t-1)}$ were linearly dependent on vectors in $V(k)$ and v_{r+j} . Any such linear dependence translates to a linear dependence of $v_{r+(b-1)t+i}$, $i \neq j$, on vectors in $V(k + b - 1)$ and $v_{r+(b-1)t+j}$. Hence the vector $v_{r+(b-1)t+j}$ must be a vector among $v_{r+(b-1)t+1}, \dots, v_{r+(b-1)t+(t-1)}$ that is not in $V(k + b - 1)$. Therefore, we can replace those positions in the last copy of u in $a * u^{*(k+b-1)} * u$ that do not correspond to $v_{r+(b-1)t+j}$ by 0; we will denote the modified vector by $a * u^{*(k+b-1)} * u^{(j)}$, where $u^{(j)}$ contains only one 1. By our discussion above, we see that

$$V(k + b) = V(a * u^{*(k+b-1)} * u^{(j)}),$$

and $\dim V(k+b) = r - k + b = r - (k - b)$. Let \mathbf{a}' be as defined in Case 1, and let $\mathbf{w}'_i = \mathbf{a} * \mathbf{u}^{*(k+b-1)} * \mathbf{u}^{(j)} * \mathbf{a}'^{*i}$. Define $W(i) = V(\mathbf{w}'_i)$. Let b' be the smallest nonnegative integer such that $W(b') = W(b' + 1)$. Then $0 \leq b' \leq k - b$. Similar to Case 1, we have

$$(i) \quad W(b') \supset I(a + (k + b + i)t, t)$$

$$(ii) \quad W(b') \supset W(b')^{2^t} \cap I(0, a + (k + b + i)t).$$

Thus, by Lemma 4.1 we have $V(\mathbf{w}'_{b'}) = W(b') = V$. The length of $\mathbf{w}'_{b'}$ is $a + (k + b)t + b't \leq a + 2kt$. Dropping the last $t - 1$ zeros in $\mathbf{w}'_{b'}$, we get a vector of length $m - a - (t + 2)$, which does not contain more than $t - 1$ consecutive 1's. So Lemma 3.8 is proved in this subcase.

Subcase 2: $V(k + b) = V$, i.e., $b = k$. Since $V(k + b) = V(\mathbf{a} * \mathbf{u}^{*(k+b-1)} * \mathbf{u}^{(j)})$ (cf. Subcase 1), we have

$$V = V(\mathbf{a} * \mathbf{u}^{*(k+b-1)} * \mathbf{u}^{(j)}).$$

Note that the binary vector $\mathbf{a} * \mathbf{u}^{*(k+b-1)} * \mathbf{u}^{(j)}$ has length $a + (k + b - 1)t + (j + 1) = m - (t + 2) - (a - j)$ and does not have more than $t - 1$ consecutive 1's. If $a \geq j$ this vector will work. So we assume that $j > a$. Recall that $\mathbf{v}_{r+1}, \dots, \mathbf{v}_{r+(j-1)} \in V(k)$ but $\mathbf{v}_{r+j} \notin V(k)$ by our choice of j . Let $(0, 1, \dots, 1) \in \mathbb{F}_2^j$ and $\mathbf{w}' = \mathbf{a} * \mathbf{u}^{*k} * (0, 1, \dots, 1)$. Then $V(\mathbf{w}') = V(k)$.

Let $\mathbf{z} = (\underbrace{1, 1, \dots, 1}_{t-j}, \underbrace{0, 1, 1, \dots, 1}_{j-1}) \in \mathbb{F}_2^t$, and let $V'(i)$ be the \mathbb{F}_{2^m} -span of $S(\mathbf{a} * \mathbf{z}^{*i})$, i.e., $V'(i) = V(\mathbf{a} * \mathbf{z}^{*i})$. Observe that when we shift the vector $\mathbf{a} * \mathbf{z}^{*k}$ to the right by j positions, we get

$$\underbrace{(0, \dots, 0)}_j * \underbrace{(1, \dots, 1)}_{t+(a-j)} * \mathbf{u}^{*(k-1)} * \underbrace{(0, 1, \dots, 1)}_j.$$

The subset represented by this vector is

$$\{\mathbf{v}_j, \mathbf{v}_{j+1}, \dots, \mathbf{v}_{r+j-1}\} \setminus \{\mathbf{v}_{a+t}, \mathbf{v}_{a+2t}, \dots, \mathbf{v}_{a+kt}\}.$$

Hence $V'(k)^{2^j} \subseteq V(k) + \sum_{i=1}^{j-1} \mathbb{F}_{2^m} \mathbf{v}_{r+i} = V(k)$. In fact, $V'(k)^{2^j} = V(k)$ since the two subspaces have the same dimension $r - k$. Similarly, $V'(k + 1)^{2^j} = V(k + 1)$. Moreover, since $\mathbf{v}_r^{2^j} = \mathbf{v}_{r+j} \notin V(k)$ we have $\mathbf{v}_r \notin V'(k)$. Hence $V'(k + 1) = V'(k) + \mathbb{F}_{2^m} \mathbf{v}_r$. It follows that $V'(k + i) = V'(k + i - 1) + \mathbb{F}_{2^m} \mathbf{v}_{r+it}$ for $1 \leq i \leq k$. In particular, $V'(2k) = V$. Since $V'(2k) = V'(2k - 1) + \mathbb{F}_{2^m} \mathbf{v}_{r+(k-1)t}$, we see that actually

$$V'(2k) = V(\mathbf{a} * \mathbf{z}^{*(2k-1)} * \underbrace{(1, 0, \dots, 0)}_t).$$

The length of the vector $\mathbf{a} * \mathbf{z}^{*(2k-1)} * \underbrace{(1, 0, \dots, 0)}_t$ is $a + (2k - 1)t + 1 = m - (t + 2) - a$. So we are also done in this subcase.

Case 3: $V(k) = V(k+1)$, i.e., $\dim V(k+1) = r - k$. In this case V is spanned by $S(\mathbf{a} * \mathbf{u}^{*k} * \mathbf{a}'^{*k})$. Thus, we have

$$\mathbf{v}_{r+1} = \sum_{i=0}^{r-1} c_i \mathbf{v}_i,$$

where $c_{a+nt} = 0$ for all $0 \leq n \leq k$. Let ℓ be the largest index such that $c_\ell \neq 0$. We consider three subcases.

Subcase 1: $\ell = a + jt + s$ with $2 \leq s \leq t-1$ and $0 \leq j \leq k-1$. Now

$$(\mathbf{v}_{r+1})^{2^{(k-j-1)t}} = \mathbf{v}_{r+1+(k-j-1)t} = \sum_{i=0}^{r-1} c_i^{2^{(k-j-1)t}} \mathbf{v}_{i+(k-j-1)t}.$$

Note that $r - t + 2 \leq \ell + (k-j-1)t \leq r - 1$. Hence $\mathbf{v}_{i+(k-j-1)t} \in S(\mathbf{a} * \mathbf{u}^{*k}) = V(k)$ for all \mathbf{v}_i with $c_i \neq 0$. Therefore we can express $\mathbf{v}_{\ell+(k-j-1)t} = \mathbf{v}_{a+(k-1)t+s}$ as a linear combination of $\mathbf{v}_{r+1+(k-j-1)t}$ and some vector in $V(k)$. It follows that V is spanned by $S(\mathbf{a} * \mathbf{u}^{*(k-1)} * \mathbf{u}^{(\ell)} * \mathbf{a}'^{*(k-j-1)} * (1, 1, 0, \dots, 0) * \mathbf{a}'^{*j})$ where $(1, 1, 0, \dots, 0) \in \mathbb{F}_2^t$ and $\mathbf{u}^{(\ell)} = (0, \underbrace{1, 1, \dots, 1}_{s-1}, 0, 1, 1, \dots, 1) \in \mathbb{F}_2^t$. For convenience, denote the vector $\mathbf{a} * \mathbf{u}^{*(k-1)} * \mathbf{u}^{(\ell)} * \mathbf{a}'^{*(k-j-1)} * (1, 1, 0, \dots, 0) * \mathbf{a}'^{*j}$ by \mathbf{z} . If $j > 0$, then we can drop $t-1$

zeros from the last copy of \mathbf{a}' in \mathbf{z} to obtain a binary vector of length $m - a - (t+2)$, which contains no more than $t-1$ consecutive 1's. If $j = 0$ we can still drop the last $t-2$ zeros from \mathbf{z} . The resulting vector has length no more than $m - (t+2)$ if $a \geq 1$. Hence we only need to consider the case $j = 0$ and $a = 0$. In that case V is spanned by $S(\mathbf{u}^{*(k-1)} * \mathbf{u}^{(\ell)} * \mathbf{a}'^{*(k-1)} * (1, 1))$ and \mathbf{v}_0 is not in the generating set. Thus, we can shift every entry in $\mathbf{u}^{*(k-1)} * \mathbf{u}^{(\ell)} * \mathbf{a}'^{*(k-1)} * (1, 1)$ to the left by one position. This still is a generating vector for V which has length $m - (t+2)$.

Subcase 2: $\ell = a + jt + 1$ with $0 \leq j \leq k-1$. If $j < k-1$, the same vector \mathbf{z} as in Subcase 1 will suit our purpose since it does not contain more than $t-1$ consecutive 1's. So we will assume $j = k-1$. We have

$$\mathbf{v}_{r+2} = \mathbf{v}_{r+1}^2 = \sum_{i=0}^{r-1} c_i^2 \mathbf{v}_{i+1}.$$

Since $\ell = a + (k-1)t + 1$, we have $c_{a+kt-1} = 0$. Note that some of the \mathbf{v}_{i+1} might be of the form \mathbf{v}_{a+nt} . However, since $\mathbf{v}_{r+2} \in V(k)$ we must have $\sum_{n=0}^{k-1} c_{a+nt-1}^2 \mathbf{v}_{a+nt} = 0$. Hence we have that $\mathbf{v}_\ell = \mathbf{v}_{a+(k-1)t+1}$ is a linear combination of \mathbf{v}_{r+2} and some vector in $V(k)$. It follows that V is spanned by $S(\mathbf{a} * \mathbf{u}^{*(k-1)} * (0, 1, 0, 1, \dots, 1) * (1, 0, 1, 0, \dots, 0) * \mathbf{a}'^{*(k-1)})$. Denote the vector $\mathbf{a} * \mathbf{u}^{*(k-1)} * (0, 1, 0, 1, \dots, 1) * (1, 0, 1, 0, \dots, 0) * \mathbf{a}'^{*(k-1)}$ by \mathbf{z}' . We see that \mathbf{z}' contains no more than $t-1$ consecutive 1's. If

$k - 1 > 0$ then we can drop the last $t - 1$ zeros of z' and obtain a vector of length $m - a - (t + 2)$. If $k - 1 = 0$ we can drop the last $t - 3$ zeros of z' . If $a \geq 2$ this vector will have length at most $m - (t + 2)$. We need to consider the case $k = 1$ and $a \leq 1$. Suppose $a = 0$ (so $r = t$). Then $v_{r+1} = c_1 v_1$ with $c_1 \neq 0$. Keep squaring both sides of this equation, we see that v_{r+1+m} is a nonzero scalar multiple of v_{r+4} . If $r > 3$ then this contradicts the fact that any r consecutive vectors in the set $\{v_0, \dots, v_{m-1}\}$ are linearly independent. So $r \leq 3$. Since $t \geq 3$ and $r = t$ we have $r \geq 3$. Thus $m = 9$. But we assumed that $m > 9$, so the case $a = 0$ cannot happen.

Now suppose $a = 1$. Then $v_{r+1} = c_0 v_0 + c_2 v_2$ and $c_0 \neq 0$, hence $v_{r+2} = c_0^2 v_1 + c_2^2 v_3$. Note that since $a = 1$, we have $V(k) = V(1) = V(\underbrace{(10 \ 11 \ \dots \ 1)}_{t-1})$. So the previous equation implies that $v_{r+2} \notin V(1)$, contradicting the assumption that $V(k+1) = V(k)$.

Subcase 3: $0 \leq \ell \leq a - 1$. Observe that $v_{r+1}, v_{r+2}, \dots, v_{r+t-1} \in V(k)$ as well. Note that $v_{r+1}^{2^{a-\ell}} = c_\ell^{2^{a-\ell}} v_a + \dots \notin V(k)$ as $v_a \notin V(k)$. It follows that $a - \ell > t - 2$. Since $a \leq t - 1$ this is only possible when $\ell = 0$. But then $v_{r+1} = c_0 v_0$, with $c_0 \neq 0$. This implies that $v_m = v_0 = c_0^{2^{r+2}+2} v_1$, which is impossible. This completes the proof. \square

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